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The propagation of short surface waves on longer gravity waves

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To understand the imaging of the sea surface by radar, it is useful to know the theoretical variations in the wavelength and steepness of short gravity waves propagated over the surface of a train of longer gravity waves of finite amplitude. Such variations may be calculated once the orbital accelerations and surface velocities in the longer waves have been accurately determined – a non-trivial computational task.

The results show that the linearized theory used previously for the longer waves is generally inadequate. The fully nonlinear theory used here indicates that for longer waves having a steepness parameter $AK = 0.4$, for example, the short-wave steepness can be increased at the crests of the longer waves by a factor of order 8, compared with its value at the mean level. (Linear theory gives a factor less than 2.)

The calculations so far reported are for free, irrotational gravity waves travelling in the same or directly opposite sense to the longer waves. However, the method of calculation could be extended without essential difficulty so as to include effects of surface tension, energy dissipation due to short-wave breaking, surface wind-drift currents, and to arbitrary angles of wave propagation.

1. Introduction

An important component of radar backscatter from the sea surface arises from the Bragg scattering. This involves surface wavelengths of the order of a few centimetres for X-band radars, or tens of centimetres for L-band. In both cases the wavelengths are usually small compared to the dominant wavelengths of ocean surface waves (10 to 10^3 m). So it becomes an important question to study how the short-wave energy is distributed with respect to the phase of the longer waves.

In the present study we shall consider the classical model of a short train of gravity waves, of small but variable steepness ak, propagated over the surface of a longer train of gravity waves of *finite* steepness AK, as in figure 1. Early workers (Longuet-Higgins & Stewart 1960) assumed that $AK \ll 1$, and in that case it was found that the variation in the wavenumber k and amplitude a of the short waves, in deep water, was given by

$$\left.\begin{aligned} \frac{k}{\bar{k}} &= 1 + AK\cos\psi + O(AK)^2, \\ \frac{a}{\bar{a}} &= 1 + AK\cos\psi + O(AK)^2, \end{aligned}\right\} \qquad (1.1)$$

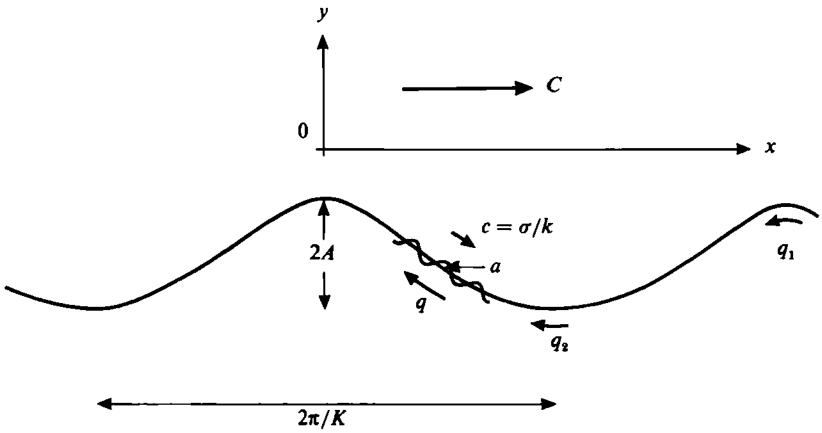

FIGURE 1. Definition sketch for short waves on long waves. The origin of y is chosen so that $q^2+2gy = 0$ on the free surface.

where $\bar{k}$ and $\bar{a}$ are the (constant) values of k and a at the mean surface level and $\psi = K(x-Ct)$ is the phase of the long waves. This gives

$$\frac{ak}{\bar{a}\bar{k}} = 1+2AK\cos\psi+O(AK)^2, \tag{1.2}$$

showing that the short waves are both shorter and steeper on the crests of the longer waves ($\psi = 2n\pi$). However, since $AK \leqslant (AK)_{\max} = 0.4432$ the maximum steepening predicted is less than 2.

Longuet-Higgins & Stewart (1960) interpreted equations (1.1) by assuming (i) that the *phase* of the short waves was conserved, i.e. that

$$kq-\sigma = \text{constant}, \tag{1.3}$$

where q is the particle speed in the long waves as seen by an observer travelling with the long-wave speed C, and σ is the intrinsic frequency of the short waves in a frame moving with speed q; next, (ii) that the intrinsic frequency σ and local wavenumber k of the short waves were related by

$$\sigma^2 = g'k, \tag{1.4}$$

where g' was the effective value of gravity for the short waves, i.e.

$$g' = g+\frac{\partial W}{\partial t}, \tag{1.5}$$

W being the vertical component of orbital velocity in the long waves; and (iii) that the short-wave energy density E was given by

$$E = \tfrac{1}{4}ga^2+\frac{\tfrac{1}{4}a^2\sigma^2}{k}, \tag{1.6}$$

representing the potential and kinetic energies respectively. The changes in short-wave energy E over the long wave could then be attributed to (a) advection by the long-wave orbital velocities, together with (b) work done by the straining of the long waves against the *radiation stress* of the short waves.

Garrett (1967) suggested that the same results (1.1) could be interpreted in terms of the conservation of *wave action*

$$N = E'/\sigma, \tag{1.7}$$

where

$$E' = \tfrac{1}{2}g'a^2 \tag{1.8}$$

is an alternative form of the short-wave energy density, and he introduced the equation

$$\frac{\partial N}{\partial t} - \frac{\partial}{\partial x}[(q-c_g)N] = 0, \tag{1.9}$$

where c_g is the group-velocity of the short waves ($c_g = \frac{1}{2}c$).

Finally, Bretherton & Garrett (1968) proved the validity of (1.9) for a general class of situations where a group of linearized short waves of wavenumber k is propagated through a *slowly varying medium* with local velocity q, under the general assumption that

$$|\nabla q| \ll kq, \quad \frac{\partial q}{\partial t} \ll \sigma q, \tag{1.10}$$

the energy density E' being defined as if the medium were locally uniform.

The great advantage of this formulation is its relative simplicity, and that there is no explicit restriction on the steepness AK of the long waves; it appears necessary to assume only that

$$ak \ll 1, \quad k \gg K. \tag{1.11}$$

In the case of AK finite, one would take as the effective (vector) gravity

$$\boldsymbol{g}' = \boldsymbol{g} - \boldsymbol{a} \tag{1.12}$$

where $\boldsymbol{a}$ is the orbital acceleration in the long wave.

This principle has been partly applied (in principle) by Phillips (1981) to calculate the variation in amplitude of short capillary–gravity waves riding on longer gravity waves. The calculation could not be carried through in detail because the effective gravity $\boldsymbol{g}'$ was not at that time known with sufficient accuracy. However, the accurate calculation of accelerations in steep gravity waves has recently been carried out by Longuet-Higgins (1985*c*), and from this it is possible to infer $\boldsymbol{g}'$ by (1.12), hence both the shortening and steepening of the short waves. In this contribution we apply the results to short gravity waves, in the first place, with application particularly to backscattering in L-band. One significant result is that for finite values of AK the short-wave steepening can actually be much greater than that given by linear theory. Moreover, it will be seen that the basic calculation of $\boldsymbol{g}'$ opens the way to the solution of other important problems, including the case when the short waves are strongly affected by capillarity.

2. Formulation of the problem

Relatively short gravity waves, of local height $2a$ and wavelength $2\pi/k$, ride on longer, progressive gravity waves of finite height $2A$ and wavelength $2\pi/K$ in deep water, where $k \gg K$ (see figure 1). It is required to find k and ak as functions of the phase of the long wave.

The intrinsic frequency σ and the wavenumber k of the short waves are assumed to be related by (1.4), where g' is the magnitude of the *effective acceleration* $\boldsymbol{g}'$ given by (1.12). Clearly, since the pressure gradient has no component tangential to the free surface, $\boldsymbol{g}'$ is always normal to the surface of the longer waves. The frequency σ and the phase-speed

$$c = \sigma/k \tag{2.1}$$

are taken as positive or negative according as the short waves travel in the same or opposite direction to the long waves. q denotes the particle speed at the surface of the longer waves, as seen in a frame of reference moving with the long-wave phase speed C. In this reference frame the long waves appear steady and the free surface is a streamline. At the mean level $y = \bar{y}$, we have $q = C$; (see Lamb 1932, p. 420).

To determine the wavenumber k at points along the surface of the long waves we assume that *the phase of the short waves is conserved*, that is

$$k(q-c) = \text{constant} = \kappa, \tag{2.2}$$

say, Hence

$$c^2 = \frac{g'}{k} = \kappa^{-1}g'(q-c) \tag{2.3}$$

or

$$c^2+\kappa^{-1}g'c-\kappa^{-1}g'q = 0. \tag{2.4}$$

This is a quadratic equation for c with solutions

$$c = -\beta \pm (\beta^2+2\beta q)^{\frac{1}{2}}, \quad \beta = \frac{g'}{2\kappa}. \tag{2.5}$$

Having found c we may calculate k from (2.3) in the form $k = g'/c^2$.

To determine the local wave amplitude a, we assume that action is conserved, that is equation (1.9). In the steady flow relative to the moving frame of reference this implies that the flux of wave action is a constant, i.e.

$$(q-c_g)\frac{E'}{\sigma} = \text{constant}, \tag{2.6}$$

where c_g the group velocity of the short waves $(=\frac{1}{2}c)$ and E' is the intrinsic energy density of the short waves, given by (1.8). Since $\sigma = g'/c$, (2.6) can also be written

$$(q-\tfrac{1}{2}c)\,ca^2 = \text{constant} \tag{2.7}$$

or

$$a \propto [(q-\tfrac{1}{2}c)\,c]^{-\frac{1}{2}} \tag{2.8}$$

(cf. Longuet-Higgins & Stewart 1961).

Finally, the local wave steepening is defined as

$$r = ak/(\bar{a}\bar{k}) \tag{2.9}$$

where a bar denotes the values at the mean level $y = \bar{y}$.

Clearly the above approach depends upon the accurate evaluation of the velocity q and the orbital acceleration $\boldsymbol{a}$ in a (long) gravity wave of finite amplitude.

3. Method of calculation

The *real*, or orbital acceleration in a surface wave must be carefully distinguished from the *apparent* accelerations as measured by a fixed vertical wave gauge (see Longuet-Higgins 1985*c*). The real accelerations, both vertical and horizontal, vary much more smoothly than the apparent accelerations, which can be very non-sinusoidal.

Numerical values of the real acceleration $\boldsymbol{a}$ were calculated by the method of Longuet-Higgins (1985*a*) which makes use of a set of quadratic relations between the Fourier coefficients a_n in Stokes's series for the Cartesian coordinates (x, y) in terms of the velocity potential. Thus if $K = g = 1$, and the free surface is given by

$$\left.\begin{aligned} y &= a_0 + \sum_1^\infty a_n \cos\left(\frac{n\phi}{c}\right), \\ x &= \frac{\phi}{c} + \sum_1^\infty a_n \sin\left(\frac{n\phi}{c}\right), \end{aligned}\right\} \tag{3.1}$$

where ϕ is the velocity potential, then the coefficients $a_0, a_1, a_2, \ldots$ satisfy the relations

$$\left.\begin{aligned} a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots &= -c^2, \\ a_1 b_0 + a_0 b_1 + a_1 b_2 + a_2 b_3 + \ldots &= 0, \\ a_2 b_0 + a_1 b_1 + a_0 b_2 + a_1 b_3 + \ldots &= 0, \end{aligned}\right\} \tag{3.2}$$

with $b_n = na_n$, $n > 0$ and $b_0 = 1$. These relations may be quickly and accurately solved for a given value of the phase-speed c (in general) or of the wave amplitude

$$A = a_1 + a_3 + a_5 + \ldots. \tag{3.3}$$

The speed q at the free surface is then found from the Bernoulli relation

$$q^2 = -2y, \tag{3.4}$$

and the vector acceleration $\boldsymbol{a}$ from the general relation

$$\boldsymbol{a} = \chi_z \chi_{zz}^* = -q^6 z_\chi^2 z_{\chi\chi}^* \tag{3.5}$$

where $z = x + \mathrm{i}y$ and $\chi = \phi + \mathrm{i}\psi$, ψ the stream function. (An asterisk denotes the complex conjugate.) In real terms this is

$$\boldsymbol{a} = -q^6 (x_\phi + \mathrm{i}y_\phi)^2 (x_{\phi\phi} - \mathrm{i}y_{\phi\phi}). \tag{3.6}$$

The effective gravity $\boldsymbol{g}'$ is then found from (1.12).

Because of the slow initial rate of convergence of the series (3.1) at high values of AK, care must be taken to include enough terms in these series. A recent study (Longuet-Higgins 1985*b*) has shown that after an initial rate of convergence like $n^{-\frac{5}{3}}$, a_n ultimately converges exponentially, the transition to exponential behaviour occurs when $n = n_c = O(\epsilon^{-3})$, where

$$\epsilon^2 = 2.0|AK - (AK)_{\max}|. \tag{3.7}$$

Since individual terms in the differentiated series for $x_{\phi\phi}$ and $y_{\phi\phi}$ at first increase like $n^{\frac{1}{3}}$, it is important, in order to ensure sufficient accuracy in the calculation, to include terms with n somewhat in excess of n_c.

Surface profile corresponding to $AK = 0.1$, 0.2, 0.3, 0.4 and the limiting value $AK = 0.4432$ are shown in figure 2. The corresponding values of the effective gravity g' are shown in figure 3. It will be seen that when $AK = 0.4$ these range from $0.65g$ at the crest of the wave $(x = 0)$ to about $1.31g$ in the wave trough.

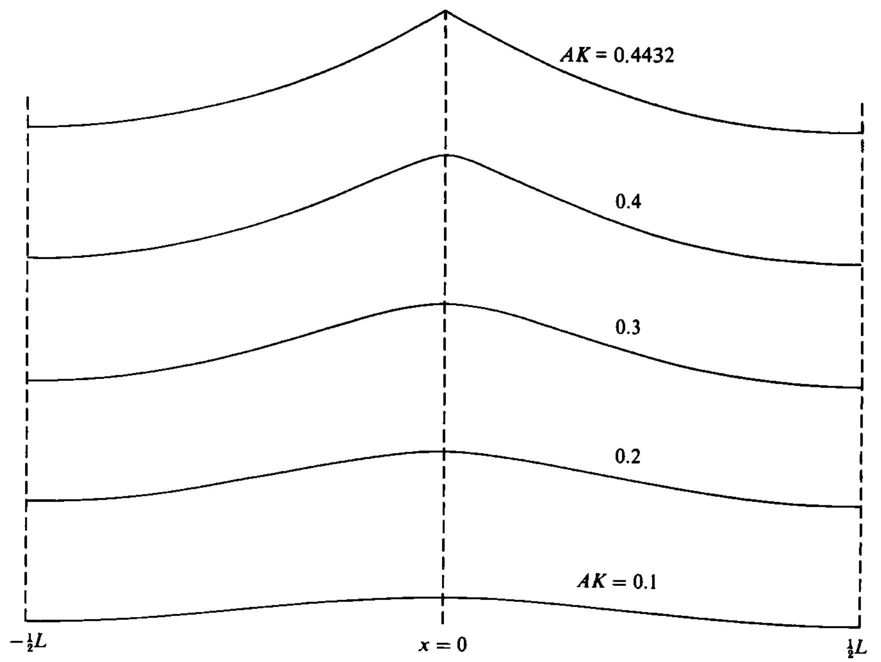

FIGURE 2. Surface profiles of gravity waves in deep water, when $AK = 0.1$, 0.2, 0.3, 0.4 and 0.4432.

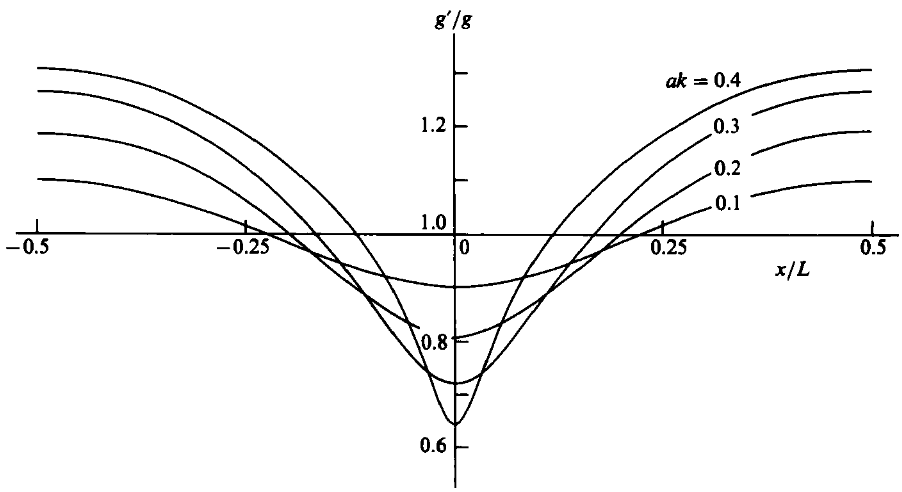

FIGURE 3. The effective value of gravity g' at the surface of steep waves, as a function of the horizontal coordinate x/L.

4. Results: variation in short-wave length

Using suffices 1 and 2 to denote values at the long-wave crest and trough respectively, figure 4 shows the relative shortening $k_1/\bar{k}$ at the crests of the long waves as compared to the mean surface level, in the three cases when $\bar{k} = 2$, 10 and 100. Similarly $k_2/\bar{k}$ shows the lengthening in the long-wave troughs. For values of AK up to 0.2 the three curves corresponding to $\bar{k} = 2$, 10 and 100 are almost indistinguishable,

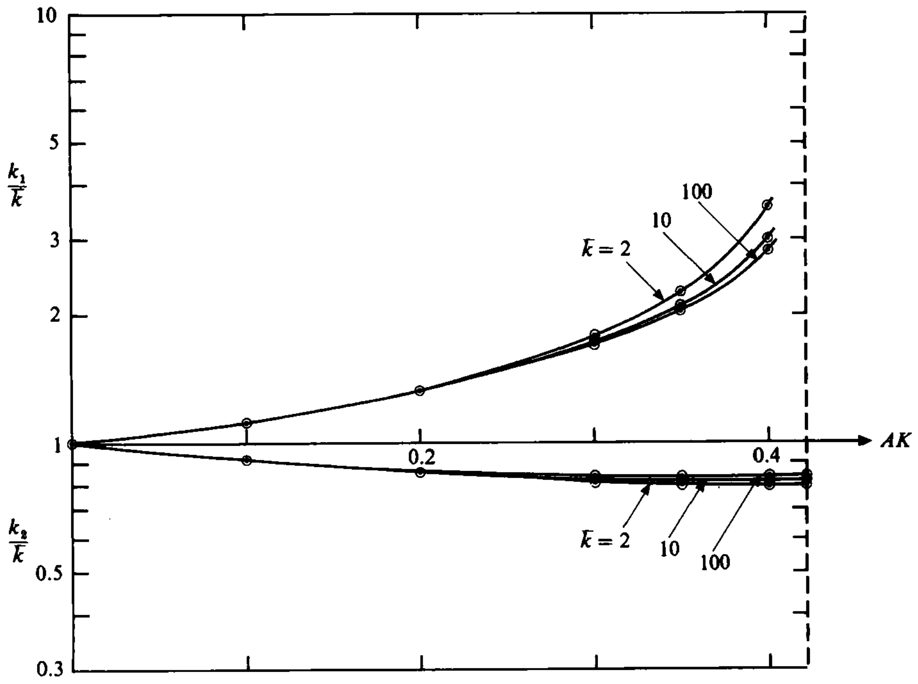

Figure 4. The relative shortening of short waves at the crests ($k_1/\bar{k}$) and in the troughs ($k_2/\bar{k}$) of long waves, as compared to the mean level, when $c > 0$. Note $K = 1$.

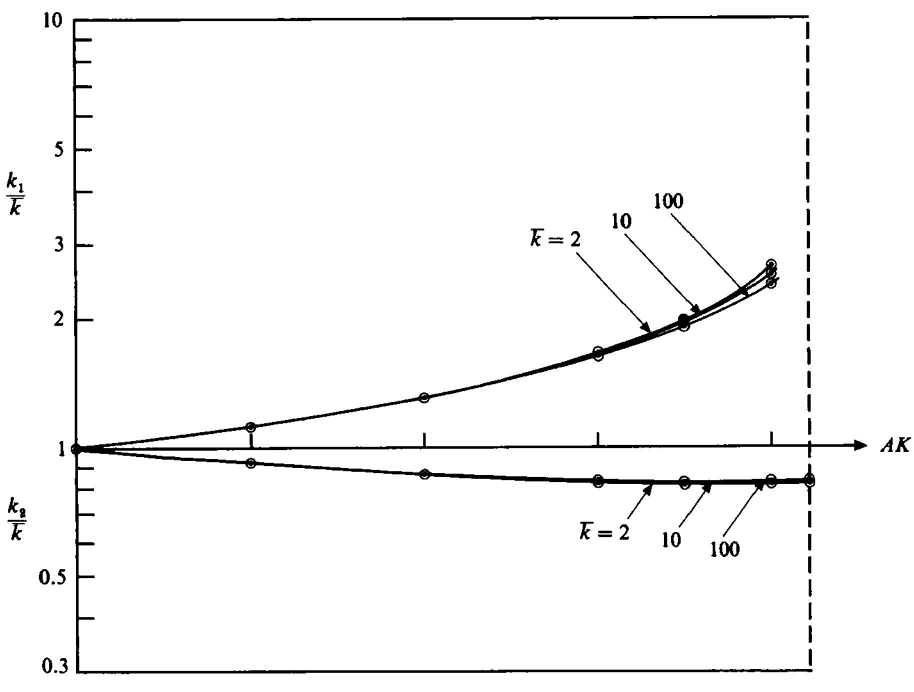

Figure 5. As figure 4, but for $c < 0$.

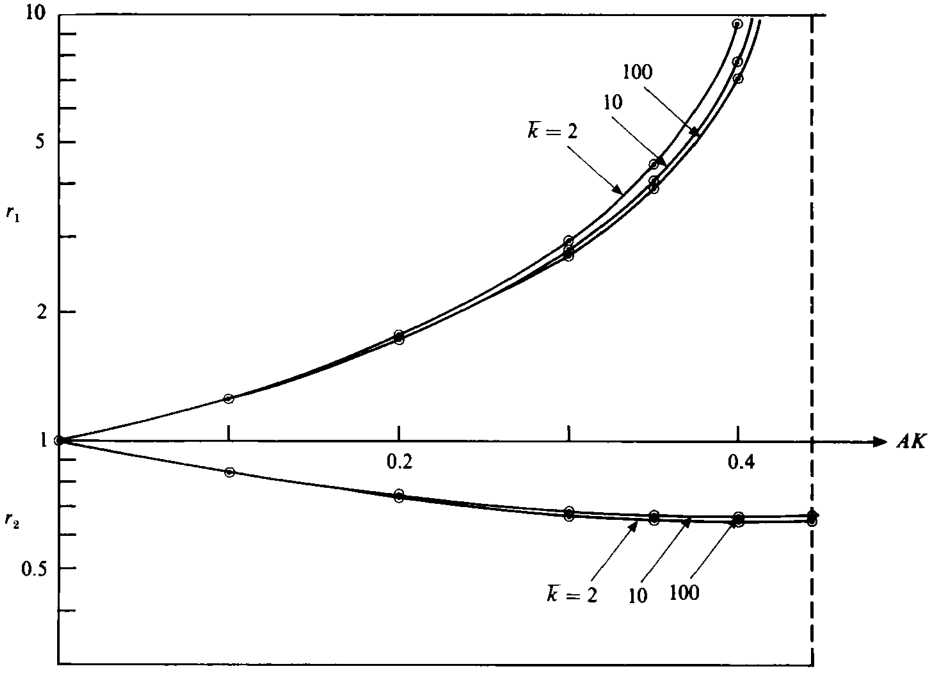

FIGURE 6. The relative steepening of short waves at the crests (r_1) and in the troughs (r_2) of long waves, as compared to the mean level, when $c > 0$.

and even when $AK = 0.4$ there is little departure from the representative curve $\bar{k} = 10$, when $k_1/\bar{k} = 3.0$ and $k_2/\bar{k} = 0.82$. Thus, the short-wave length varies over a range of about $3\frac{1}{2}$ to 1. This is for $c > 0$, when the short waves travel in the same sense as the longer waves. Figure 5 shows a similar plot when $c < 0$, and the short waves travel in the opposite sense. Here the variation in k is only slightly less. However, as $AK \rightarrow (AK)_{\max}$ it can be shown that $k_1/\bar{k} \rightarrow \infty$ when $c > 0$, but remains finite when $c < 0$.

5. Variation in the wave steepness

Figures 6 and 7 show the variation in steepening r of the shorter waves, in a similar manner to figures 4 and 5. Thus

$$r_1 = \frac{a_1 k_1}{\bar{a}\bar{k}}, \quad r_2 = \frac{a_2 k_2}{\bar{a}\bar{k}}. \tag{5.1}$$

Again the three curves corresponding to $\bar{k} = 2$, 10 and 100 lie very close together, showing that not only the wavelength variation but also the steepness variation is practically independent of short-wave length.

When $AK = 0.4$, however, the short-wave steepness may vary by a factor of as much as 8 between the long-wave crests and the mean level. This compares with a factor less than 2 given by linear theory.

The variation of steepness r over the profile of the long waves is shown in figure 8 as a function of x/L, and for different values of AK, using the representative short wavenumber $\bar{k} = 8$. Comparing $AK = 0.4$ with $AK = 0.1$, one sees the distorting effect of nonlinearity in the long waves.

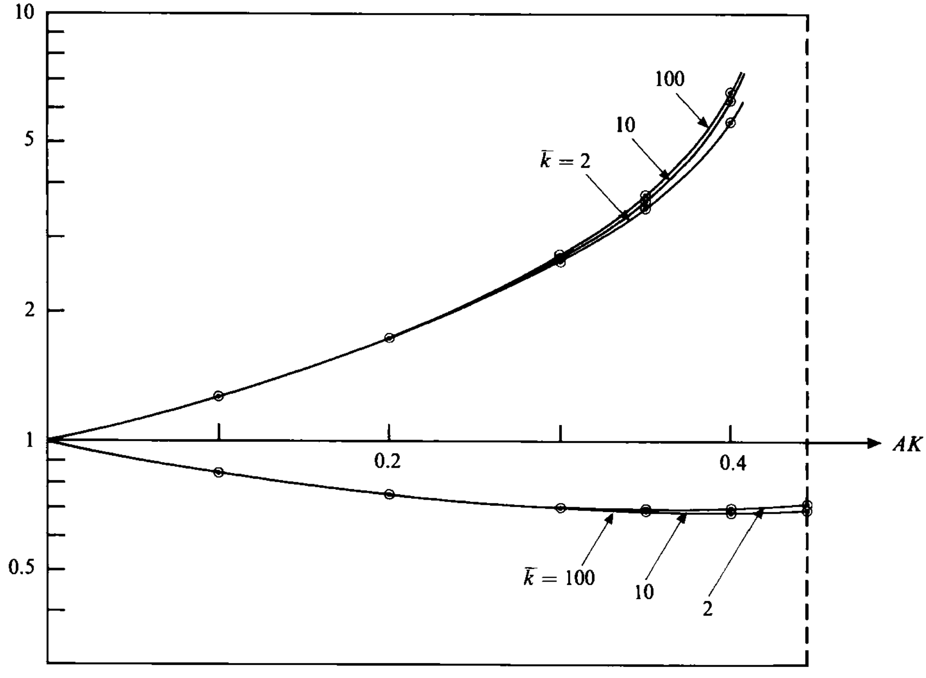

FIGURE 7. As figure 6, but for $c < 0$.

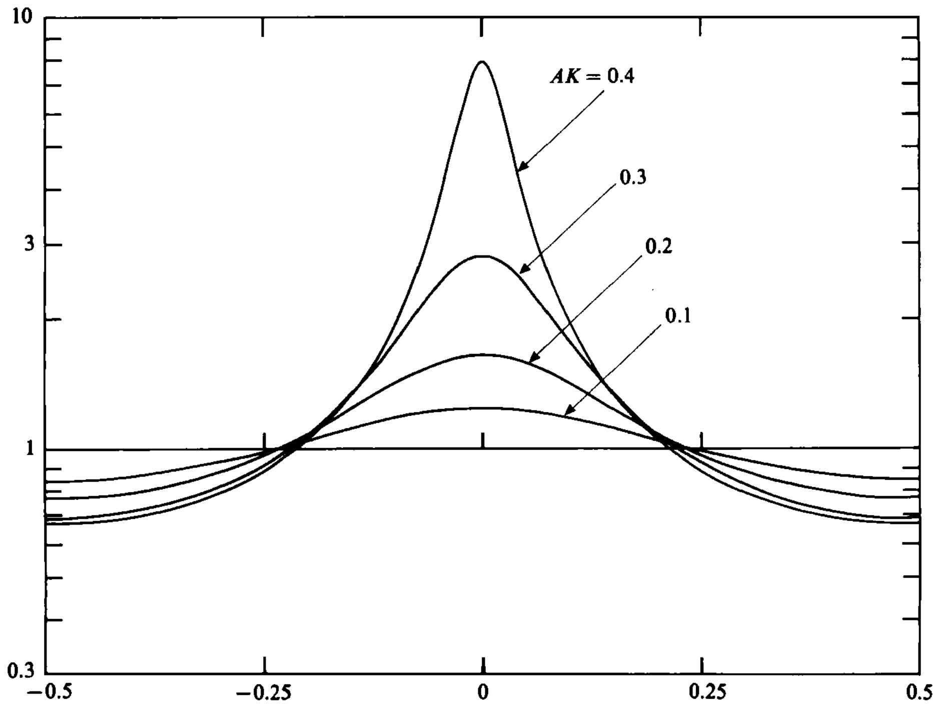

FIGURE 8. The relative steepening r as a function of the horizontal coordinate x/L, when $\bar{k} = 8$, $c > 0$.

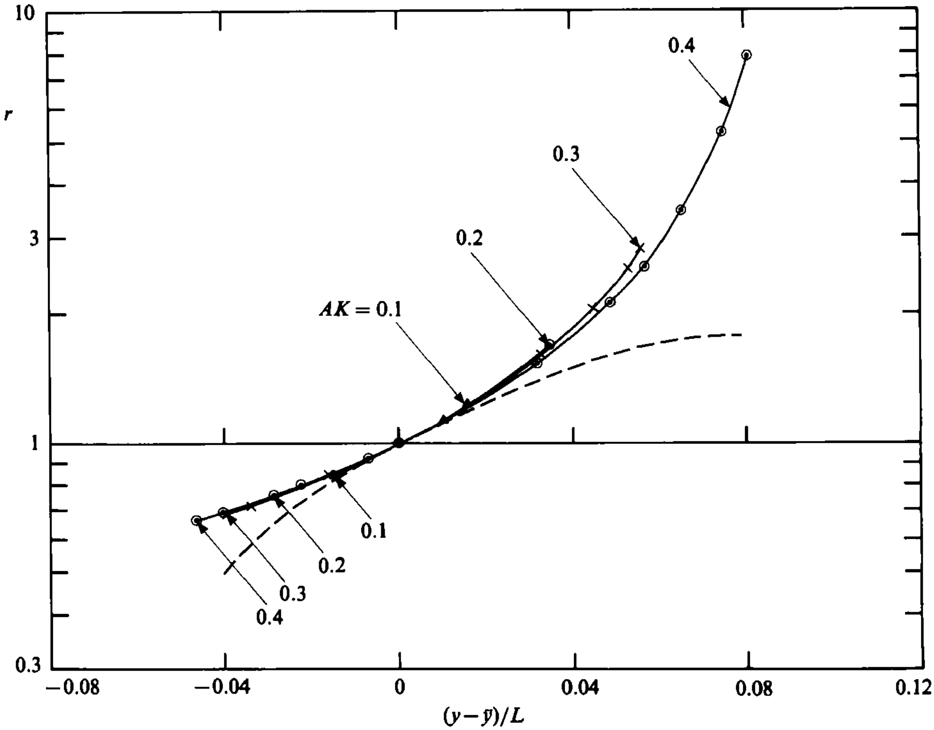

FIGURE 9. The relative steepening r as a function of the vertical coordinate $(y-\bar{y})/L$ when $\bar{k} = 8$, $c > 0$.

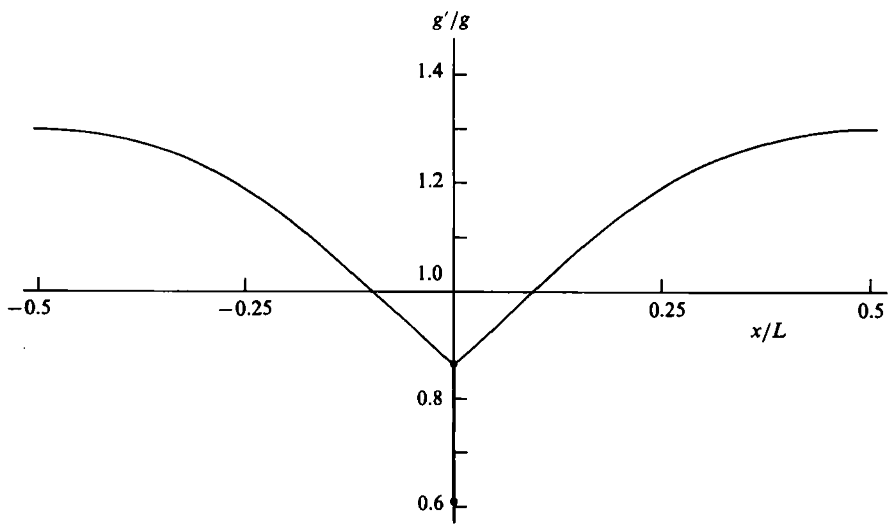

FIGURE 10. The effective value of gravity g' at the surface of deep-water waves of limiting steepness.

Finally, in figure 9 the three curves of figure 8 are plotted instead against $(y-\bar{y})/L$, that is the vertical height above the mean level $\bar{y}$. It now appears that all the curves collapse almost onto a single curve. This property may be useful in approximate analytical work. The appropriate nonlinear steepening is quite different from the linear theory, shown in figure 9 by the broken curve.

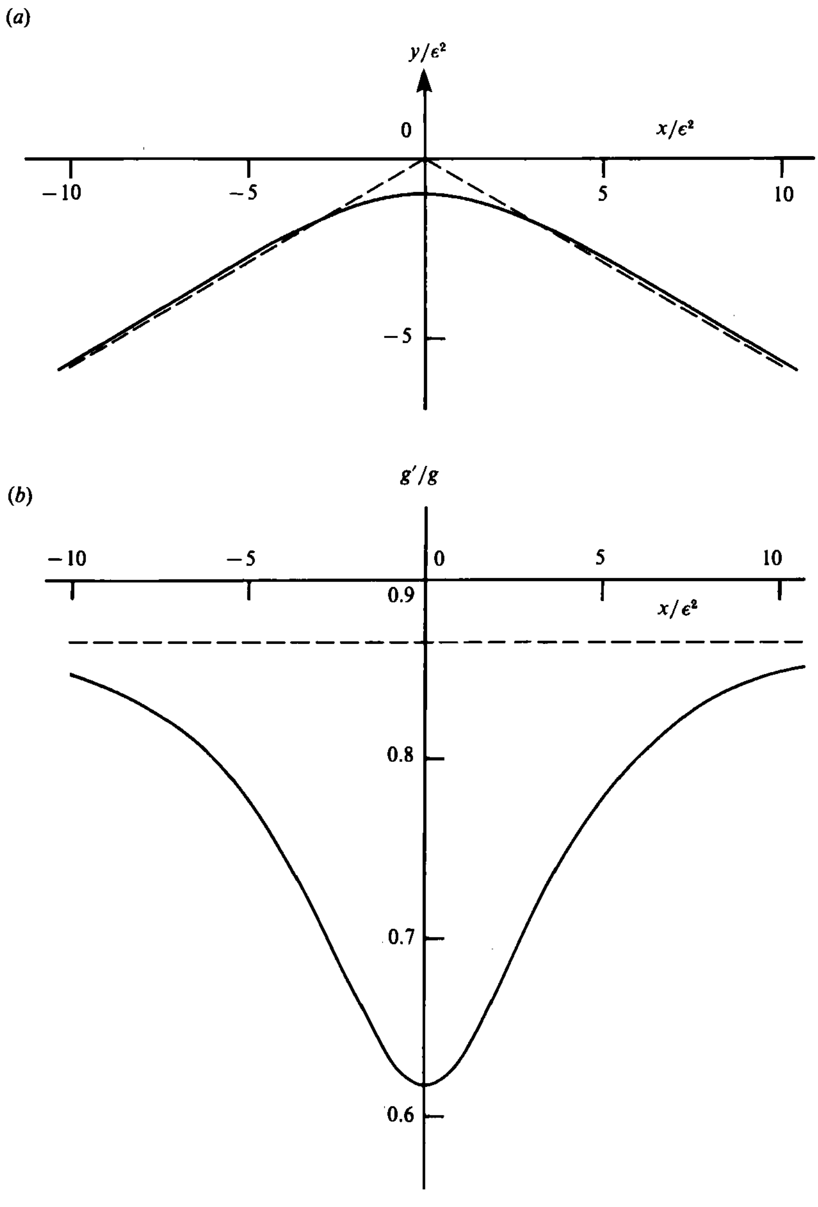

FIGURE 11. (*a*) The surface profile near the crest of a steep gravity wave, scaled according to $\epsilon^2 = q_1^2/2C$. (*b*) The effective value of gravity g' near the crest of a steep gravity wave.

6. Limiting waves

Our previous calculations have been carried only as far as $AK = 0.4$. In this Section we shall consider the limiting behaviour of the solutions as $AK \rightarrow (AK)_{\max}$, and the validity of the present approximations at large wave steepnesses.

Consider first the effective acceleration g' in a limiting wave. In figure 10 g'/g is plotted against the horizontal coordinate x/L, using the computations by Williams (1981, Table 12*d*). In the wave troughs, $g'/g = 1.301$ and near the crests, since the surface slope tends to 30°, we have $g'/g = 3^{\frac{1}{2}}/2 = 0.866$. At the crest itself, however, the downwards acceleration tends to $0.388\,g$ (Williams 1981), so that $g'/g = 0.612$.

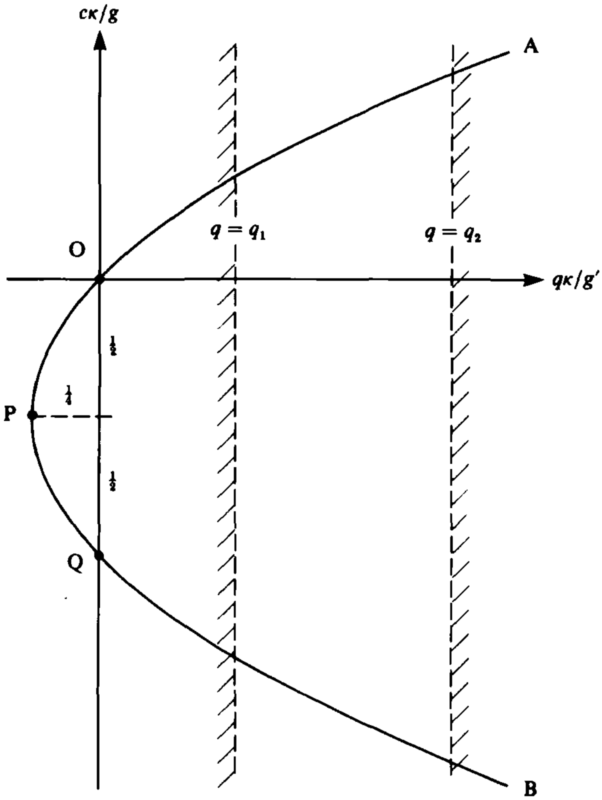

FIGURE 12. Graph to illustrate the behaviour of the phase-speed c, as given by (2.4).

For near-limiting waves, the behaviour near the crest is given by the theory of the almost-highest wave (Longuet-Higgins & Fox 1977, 1978), which is valid when

$$q^2/C^2 = 2\epsilon^2 \ll 1. \tag{6.1}$$

In this approach we introduce scaled coordinates $z_s = z/\epsilon^2$, where $z = x+iy$, and a scaled velocity potential $\chi_s = \chi/\epsilon^3$ where $\chi = \phi + i\psi$. Making use of the numerical coordinates of the free surface as given in table 3 of Longuet-Higgins & Fox (1977) we can easily calculate the radius of curvature R and hence the normal component of the particle acceleration

$$a_N = q^2/R = 2gy/R \tag{6.2}$$

at each point on the surface and hence the value of g'. This is shown in figure 11(b), where g'/g is plotted against x/ϵ^2. For comparison with the steepest wave in figure 2, (3.7) shows that when $AK = 0.40$, then $\epsilon^2 = 0.086$.

Now consider the propagation of short waves near the crest of a fairly steep longer wave. The values of the phase speed c for given values of q and κ/g' are shown in figure 12, $c\kappa/g'$ being plotted against $q\kappa/g'$ according to (2.4). Since q is always positive, the two roots (2.5) correspond to the branches OA and QB of the parabola respectively. In fact the relevant sectors of the parabola are those lying between $q = q_1$ and $q = q_2$. For moderate wave steepnesses, q_1 and q_2 are of order C while $\kappa = \bar{k}(C-\bar{c})$ is of order $\bar{k}C$ when $\bar{k} \gg 1$. Hence $q\kappa/g'$ is generally large.

However, when the longer waves become steep, we have $\epsilon \to 0$, hence $q/C \to 0$. For any finite value of $\bar{k}$, suppose it were possible for the left-hand boundary in figure 12 to approach the axis $q\kappa/g' = 0$. Then the positive root of (2.4) would give $c \sim q$, and so from (2.3) $k \sim g'/q^2$, independently of $\bar{k}$. In other words the local wavelength $2\pi/k$ of the short waves would be of order ϵ^2, comparable to the radius of curvature of the crest. Hence the short-wave approximation would not be applicable. Similar considerations apply even more strongly to the negative root of (2.4).

For the short-wave approximation to remain valid we must have $k\epsilon^2 \gg 1$. But since $k = g'/c^2$ and $\epsilon^2 = q^2/2C^2$ this implies $q^2 \gg c^2$. Hence q/c is at least moderately large, and the region of interest in figure 12 lies well to the right, where $q\kappa/g' \gg 1$. This in turn means that we must have $\epsilon^2\bar{k}^2 \gg 1$. For example, when $AK = 0.4$ then $\bar{k}^2 \gg 10$. Thus in figures 4–7 only the plots corresponding to $\bar{k} = 10$ and 100 are quantitatively valid, at this value of AK.

Nevertheless some qualitative conclusions may be drawn. From figure 12 it is clear that the phase speed c and hence the lengthscale k^{-1} is always greater for oppositely travelling short waves approaching the long-wave crest than it is for short waves travelling in the positive direction. This suggests that there may be a real distinction between 'spilling' and 'plunging' breakers, the former being caused by forwards-travelling short-wave energy, and the latter by perturbations travelling in the opposite sense.

7. Conclusions

We have shown that by taking full account of the nonlinearity of the longer waves and by using the principle of action conservation for the shorter waves one can calculate accurately the short-wave steepening. This can be several times greater than that predicted by linear theory. The short-wave approximation cannot, however, be extended to long waves of limiting, or near-limiting, steepness.

We note that according to our assumptions in (1.1), the principle of action conservation is expected to be only an approximation. To test this principle, we have studied in another paper (Dysthe *et al.* 1987) a simple model in which the governing equations are ordinary differential equations capable of exact integration by numerical methods. The model suggests that action for the shorter waves is indeed conserved closely, though not precisely.

All the results of the present paper depend upon a precise calculation of the local gravity g'. Hence we have considered only the case when the short waves are pure gravity waves. However, it must be realized that the basic calculation of g' for the long waves opens the way to a solution of other important problems, particularly the case when the short waves are capillary or capillary–gravity waves. A more general treatment is in progress which includes the dissipation of the short waves by breaking and the regeneration of the short waves by the wind.

Most of the calculations contained in this paper were first presented in a report to the TOWARD Hydrodynamics Committee at the Naval Research Laboratory, Washington D.C. in October 1985. The report was prepared at the Cal. Tech. Jet Propulsion Laboratory, Pasadena, with the kind cooperation of Dr C. Elachi and Dr O. H. Shemdin.

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